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Department of Mathematics
M.Sc.-II(Semester-IV) MATHEMATICS
Functional Analysis (MMT 401)
Subject Code: 96116

Question bank

Que.1. Define the following term

1) Banach space
2)Complete metric space
3) Positive operator
4)Bounded linear transformation
5) Conjugate operator
6) Unitary space
7)Coset
8) Poset
9) Reflexive space
10) Self adjoint operator
11)Inner product space
12) Linear Space
13) Direct sum
14) Normed linear space
15) Basis for linear space
16) Quotient space
17)Euclidean space
18) Comparable elements
19) Null set
20) Self adjoint operator
21)Linear Transformation
22)Linear sum
23) Equivalent norms
24) Orthogonal complement of a set
25) Metric space
26) Second conjugate space
27)Subspace
28)Linear span
29) Hilbert space
30) Dimension of linear space
16)Finite dimensional normed linear space
33) Separable space

Que. 2. Fill in the blanks.

1) A complete normed linear space is called ------.
2) Every non-zero Hilbert space has ________ set.

3) The dimension of set of all mxn matrices is _______.
4) If S_1 and S_2 are subsets of Hilbert space H and $S_1 \subseteq S_2$ then _____.

5) Let H be a Hilbert space and $H = \{0\}$ then H has __________ orthonormal set.

6) A normed linear space N is said to be reflexive if N is isometrically isomorphism onto ------.
7) A complex Banach space in which norm arises from inner product is called as ------- space.

8) A non-empty subset C of Hilbert space H is said to be convex if ________.

9) For any $x \in H,\ (x, 0) =$______.

10) An operator on Hilbert space H is said to be self-adjoint then T = -------.
11) Codomain of functional is ------.
12) In Hilbert space H, $\{0\}^{\perp}$ = ____.
13) Let N be real normed linear space and $f \in N^*$. If $f(x) = 0$ then $x =$________.

14) Any two norms on -------- dimensional normed linear space are equivalent.
15) A closed linear subspace of a reflexive Banach space is ------.
16) For any $y \in H,\ (0, y) =$______.

17) ___________ is the only vector which is orthogonal to itself.

18) Let H be a Hilbert space then $H^{\perp}$ = ____.
19) The Cauchy Schwarz inequality is _____.

20) Let $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$ be n ordered touples of real or complex numbers. The inequality

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$$ is called ______.

21) A self-adjoint operator A is said to be positive operator if ____ .

22) Let H be a Hilbert space, for $x, y \in H$, $(x, \alpha y) =$ ______.

23) For any $x, y \in H$, the parallelogram low is______________ .

24) A complete normed linear space is called ____

25) Let $(P, \leq)$ be a poset. Every chain has upper bound then P has ___ element.

26) The Minkowski inequality is _____.

27) The dimension of set of all mxn matrices is___________.

28) Let $x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$ be n ordered touples of real or complex numbers. The inequality

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

Where p>1, q>1 and $\frac{1}{p}+\frac{1}{q}=1$ is called __________ .

29) The Holder's inequality is _______ .

30) Let $x=(x_1,x_2,x_3,\dots.,x_n)$ and $y=(y_1,y_2,y_3,\dots.,y_n)$ be n ordered touples of real or complex numbers. The inequality

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

where $p \ge 1$ is called_______ .

31) Let H be a Hilbert space For $x,y \in H,$ *the inequality* $|(x,y)| \le \| x \| \| y \|$ is called_______ inequality.

Que.3. Solve the following questions. (Short answer)

1) Let T be an operator on normed linear space N and T^* be its conjugate operator on N^* then show that $\|T^*\| = \|T\|$.

2) $\|.\|_1$ and $\|.\|_2$ be equivalent norms on normed linear space N then show that $x_n \to x$ with respect to $\|.\|_1$ iff $x_n \to x$ with respect to $\|.\|_2$.

3) If N is reflexive space then show that every $f \in N^*$ attends its norm on unit sphere.

4) Let T be an operator on normed linear space N and T^* be its conjugate operator on N^* defined by $T^*(f(x)) = f(T(x))$, where $f \in N^*$ and $x \in N$.
Define $\emptyset$: B(N) → B(N*) by $\emptyset(T) = T^*$ then show that $\emptyset$ preserves the identity and $(T^*)^{-1} = (T^{-1})^*$.

5) Prove that in Hilbert space, inner product is jointly continuous.

6) Show that an operator T on Hilbert space H is self-adjoint iff (T_x, x) is real for $x \in$ H.

7) If M is closed linear subspace of Hilbert space H then prove that $H=M\oplus M^{\perp}$.

8) Let T be an arbitrary on Hilbert space H then show that T^*T and TT^* are positive operators.

9) Let T be an arbitrary on Hilbert space H then show that T^*T and TT^* are positive operators.

10) Let S_1 and S_2 are subset of Hilbert space H and $S_1 \subseteq S_2$ then show that ${S_2}^{\perp} \subseteq {S_1}^{\perp}$.

11) Let H be a Hilbert space. For any $x, y \in H$ show that $(x, y) = Re(x, y) + i\, Re(x, iy)$

12) Let N be a normed linear space T and T' be in $B(N)$ then show that $TT' \in B(N)$.

13) Show that every normed linear space is metric space.

14) Let T be an operator on normed linear space N and T^* be its conjugate operator on N^* defined by $T^*\big(f(x)\big) = f\big(T(x)\big)$, where $f \in N^*$ and $x \in N$. Define $\emptyset\colon \mathrm{B(N)} \to \mathrm{B(N^*)}$ by $\emptyset(T) = T^*$ then show that $\emptyset$ preserves the identity and $(T^*)^{-1} = (T^{-1})^*$.

15) Let T be an operator on normed linear space N and T^* be its conjugate operator on N^* then show that T^* is linear.

16) Let l_2^n denotes linear space of all n-tuple scalars. For $x = (x_1, x_2, \dots \dots, x_n)$, $y = (y_1, y_2, \dots \dots, y_n) \in l_2^n$ define $(x, y) = \sum_{i=1}^{n} x_i\, \bar{y}_i$. Show that l_2^n is Hilbert space.

17) Let H be a Hilbert space. For any $x, y \in H$ show that $\| x + y\|^2 + \| x - y\|^2 = 2 \| x\|^2 + 2 \| y\|^2$

18) Show that the orthonormal set in Hilbert space H is linearly independent.

19) Let T be an arbitrary on Hilbert space H then show that $\mathrm{T^*T}$ and $\mathrm{TT^*}$ are positive operators.

20) Show that an operator T on Hilbert space H is self-adjoint iff (T_x, x) is real for $x \in$ H.

21) Let N and N' be the normed linear spaces and $T : N \to N'$ be a linear transformation then show that T is continuous on N iff T is continuous at $0 \in N$.

22) Let N and N' be the normed linear spaces over the same field K and $T : N \to N'$ be a transformation then show that null set of T i.e. N(T) is closed when T is continuous.

23) Let (B, || . ||) be a Banach space which is a direct sum of linear subspaces M and N (i.e. B = M⊕N) so that $z \in B$ can be uniquely expressed as $z = x + y$ where $x \in M$ and $y \in N$. Then show that B is normed linear space with respect to new norm defined by $\|z\|' = \|x\| + \|y\|$.

24) Let T be an operator on normed linear space N and T^* be its conjugate operator on N^* then show that T^* is linear.

25) Show that the orthonormal set in Hilbert space H is linearly independent.

26) Let l_2^∞ denotes linear space of all sequences such that $\sum_{i=1}^{\infty}|x_i|^2 < \infty$. For $x = (x_1, x_2, \dots \dots, x_n, \dots)$, $y = (y_1, y_2, \dots \dots, y_n, \dots) \in l_2^\infty$ define $(x, y) = \sum_{i=1}^{\infty} x_i\, \bar{y}_i$. Show that l_2^∞ is Hilbert space.

27) Let H be Hilbert space and for $x, y \in H$, show that $|(x, y)| \leq \|x\|\ \|y\|$.

28) Show that the orthonormal set in Hilbert space H is linearly independent.

29) Show that an operator T on Hilbert space H is self-adjoint iff (T_x, x) is real for $x \in$ H.

30) Every convergence sequence in normed linear space N is a Cauchy sequence.

31) Show that the real linear space $\mathbb{R}$ with $\| x \| = |x|$, $(x \in \mathbb{R})$ is Banach space.

32) Let $T_1, T_2 \in B(N, N')$the show that $T_1 + T_2 \in B(N, N')$.

33) Let M be a closed linear subspace of normed linear space N and $T: \frac{N}{M} \to N$ defined as $T(x + M) = x$ then show that T is linear.

34) Show that in normed linear space norm is a continuous function.

35) $\|.\|_1$ and $\|.\|_2$ be equivalent norms on normed linear space N then show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence with respect to $\|.\|_1$ iff $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence with respect to $\|.\|_2$.

36) Let T: M →N be a linear transformation. If M is finite dimensional normed linear space then show that T is bounded.

37) Let N be a normed linear space. If N* is separable space then show that N is also separable.

38) Show that $B(N, N')$ is closed under scalar multiplication.

39) Show that in normed linear space addition is jointly continuous.

40) If M is a finite dimensional subspace of normed linear space N then for a real number a with $0 < a < 1$ show that their exist $x \in N$ with $d(x, M) = 1$.

41) Show that if N is reflexive then N is complete.

42) Show that i) $(x, \alpha y) = \bar{\alpha}\ (x, y)$ ii) $(\alpha x - \beta y) = \alpha(x, z) - \beta(y, z)$

iii) $(x, \alpha y + \beta z) = \bar{\alpha}\ (x, y) + \bar{\beta}\ (x, z)$

43) Let H be a Hilbert space. For any $x, y \in H$ show that $4(x, y) = \| x + y\|^2 - \| x - y\|^2 + i \| x + iy\|^2 - i\| x - iy\|^2$

44) Let H be a Hilbert space. For any $x, y \in H$ show that $\| x+y\|^2 - \| x-y\|^2 = 4\, Re(x, y)$

45) Show that in normed linear space multiplication is jointly continuous.

46) Let H be a Hilbert space then show that i) $H^{\perp} = \{0\}$ ii) $\{0\}^{\perp} = H$

47) Let S be a subset of Hilbert space H then show that i) $S \cap S^{\perp} \subseteq \{0\}$

ii) $S \subseteq S^{\perp\perp}$

Que.4 Solve the following questions (long answers).

1) Let N be the normed linear space and $x_0 \neq 0$ is a vector in N then show that there exist a functional $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.
2) 10) If M is closed linear subspace of normed linear space N and $x_0 \notin M$ then show that there exist a functional $f_0 \in N^*$ such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.
3) Let $\{e_i\}$ is an orthonormal set in H and x is any vector in H then show that the set $S = \{e_i \mid (x, e_i) \neq 0\}$ is either empty or countable.
4) Let T be an operator on normed linear space N and T^* be its conjugate operator on N^* defined by $T^*\big(f(x)\big) = f\big(T(x)\big)$, where $f \in N^*$ and $x \in N$.
   Define $\emptyset: B(N) \to B(N^*)$ by $\emptyset(T) = T^*$ then show that
   (i) $\emptyset$ is isometrically isomorphism
   (ii) $\emptyset$ preserve the norm and reverse the product.
5) Let H be a Hilbert space. Show that the set of all self-adjoint operators in B(H) is closed real linear subspace of B(H).
6) If A is positive operator on H then show that I+A is non-singular.
7) 12) Show that a closed convex subset of Hilbert space H contains a unique vector with smallest norm.

8) Let S be a non-empty subset of Hilbert space H then show that $S^{\perp}$ is closed linear subspace of H.

9) Show that N is finite dimensional normed linear space iff the set S = $\{x \in N \mid \|x\| \leq 1\}$ is compact.
10) Let B and B' be Banach spaces and $T: B \to B'$ is linear map. Show that if T is continuous then the graph of T i.e. G_T is closed set.
11) Show that non-empty subset X of normed linear space N is bounded iff $f(x)$ is bounded set of real numbers for $f \in N^*$.
12) Let C[a, b] be a class of all continuous real valued functions on [a, b]. Define $\|f\| = max\{|f(t)|$ $such\ that\ t\epsilon[a, b]\}$. Show that C[a, b] is Banach space with respect to given norm.
13) Show that any two norm on a finite dimensional normed linear space N are equivalent.

14) Let H be a Hilbert space For $x, y \in H$ show that $|(x, y)| \leq \| x \| \, \| y \|$.

15) Let M be a proper closed subspace of normed linear space N. Show that for every real number a with $0 < a < 1$ there exist a point $x_a \in N$ such that $\|x_a\|$ =1 and $d(x_a$,M) = $inf\{\|x_a - m\| \ | \ m \in M\} \geq a$.

16) Let M be a closed linear subspace of Hilbert space H and x be a vector not in M. If $d = d(x, M)$ then show that there exist a unique vector $y_0 \in M$ such that $\|x - y_0\| = d$.

17) Let N be a normed linear space. Show that for each vector $x \epsilon N$ induces a functional F_x on N^* defined by $F_x(f) = f(x)$ such that $\|F_x\| = \|x\|$. Further show that the mapping $T : N \to N^{**}$ such that $T(x) = F_x$ is isometric isomorphism on N into N^{**}.

18) Let $\{e_1, \ e_2, \ldots\ldots, e_n\}$ be the finite orthonormal set in H then for any vector x in H show that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x\|^2$.

19) Let M is a closed proper subset of Hilbert space H then show that there exist a non-zero vector $z_0 \in H$ such that $z_0 \perp M$.

20) Show that a closed convex subset of Hilbert space H contains a unique vector with smallest norm.

21) Let $\{e_i\}$ be an orthonormal set in Hilbert space H then show that following statements are equivalent

(i) $\{e_i\}$ is complete (ii) $x \perp \{e_i\} \Longrightarrow x = 0$

(iii) $x = \sum_i (x, e_i)e_i$, $(x \in H)$ (iv) $\| x\|^2 = \sum_i |(x, e_i)|^2$, $(x \in H)$

22) Let M be a linear subspace of normed linear space N and f be a continuous linear functional defined on M. For $x_0 \notin M$ if $M_0 = [M \cup \{x_0\}]$ spanned by M and $\{x_0\}$ then show that f can be extended to a continuous linear functional f_0 on M_0 such that $\|f_0\| = \|f\|$.

23) Let B be a complex Banach space in which parallelogram satisfy and inner product is given by $4(x, y) = \| x + y\|^2 - \| x - y\|^2 + i \| x + iy\|^2 - i\| x - iy\|^2$ then show that B is Hilbert space.

24) Let N be a finite dimensional normed linear space and dim(N)=n. Prove that every linear functional on N is continuous and dim(N^*) = dim(N).

25) Let M be closed linear subspace of norm linear space N. If norm of coset $x + M$ in the quotient space $\frac{N}{M}$ is defined by $\|x + M\| = \inf\{\|x + m\| \ | \ m \in M\}$ then prove that $\frac{N}{M}$ is norm linear space. Further if N is the Banach space then prove that $\frac{N}{M}$ is also a Banach space.

26) Let N and N′ be the normed linear spaces and $T: N \to N'$ be a linear transformation. Show that T is bounded iff T is continuous.

27) Show that a normed linear space N is complete iff the set $S = \{x \in N \ such \ that \ \| x \| = 1\}$ is complete.

28) Show that every finite dimensional normed linear space is reflexive.

29) Let $\{e_i\}$ be an orthonormal set in Hilbert space H then show that following statements are equivalent

(i) $\{e_i\}$ is complete (ii) $x \perp \{e_i\} \Longrightarrow x = 0$

(iii) $x = \sum_i (x, e_i)e_i$, $(x \in H)$ (iv) $\| x\|^2 = \sum_i |(x, e_i)|^2$, $(x \in H)$.

30) Prove that Euclidean space is Banach space with respect to Euclidean norm.

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1)Define following terms.

1) Graph isomorphism
2) Complete graph
3) Incidence matrix
4) Complete lattice
5) Generating function
6) Regular graph
7) Spanning subgraph
8) Adjacency matrix
9) Hasse diagram of poset
10) Discrete numeric function
11)Self complementary graph
12) Bipartite graph
13) Unicyclic graph
14) Consistent Enumeration
15) Induced subgraphs
16) Intersection of two graphs
17) Path
18) Trail
19) Component
20) Wheel graph
21) Minimal element of a poset
22) Similarity mapping
23) Well ordered set
24) Lattices
25) Distributive lattice.

2) Fill in the blanks.

1) A simple graph in which each pair of distinct vertices is join by an edge is called…
2) A Complete graph K_7 has …number of edges.
3) Let G be the graph with n vertices has exactly one even vertex then odd vertices in $\bar{G}$ is…

4) Radius of Peterson graph is …

5) The smallest integer n such that the complete graph K_n has atleast 500 edges is…

6) A complete graph K_n is …regular.

7) If the graph G is simple then all the entries of A(G) on main diagonal are …

8) The Sum of the elements in each column of incidence matrix is …

9) A graph which contains no cycle is called …

10) A graph which is connected and acyclic is called …

11) If G has 17 edges, then maximum possible vertices in G are…

12) If G has 21 vertices, then minimum possible no. of edges in G are…

13) A graph G which is connected and contains precisely one cycle is called …

14) Let A & B be the two finite sets then $|A - B| = \cdots$

15) Among the integers 1 to 300 the no. of integers divisible by 3 but nor by 5 is …

16) The minimum number of students in a class sure that 3 of them are born in the same months are …

17) Let L be a complemented lattice, with unique complements then join irreducible elements other than 0 are its …….

18) Consider the Numeric function $a_r = 1\ ;\ r \geq 0$ then corresponding generating function is……….

19) Consider the Numeric function $a_r = 2\ ;\ r \geq 0$ then corresponding generating function is……….

20) If the numeric function b=αa Where α is constant then $B(z) = \cdots$

21) Let $b_r = 2r + 1\ ;\ r \geq 0$ then $B(z) = \cdots$

22) The generating function for discrete numeric function 1,0,1,0,1,0, ... is...

23) An element $'a'$ which immediately succeed $'0'$ is called…

24) A lattice L is said to be complemented if…

25) Example of non-distributive lattice is…

3) Short answer questions:

1) Let G be a graph with n vertices, out of which t have degree k and other have degree $k + 1$ then prove that $t = (k + 1)n - 2e$.

2) Prove that if u is an odd vertex of the graph G then there must be a path in G from u to another odd vertex v in G.

3) Prove that in any graph G there is an even number of odd vertices.

4) Prove that the set of any seven distinct integers, there must exist two integers in this set whose sum or difference is multiple of 10.

5) Let G be an acyclic graph with n number of vertices and k number of connected components then show that G has $n - k$ edges.

6) Give an example of graph in which the length of longest cycle is 9 and

the length of the shortest cycle is 4.

7) Let G be a simple graph with n vertices where $n \geq 2$. Prove that G has 2 vertices u and v with $d(u) - d(v)$.

8) Let G be a graph with n vertices and exactly $(n - 1)$ edges. Prove that G has Either vertex of degree 1 or an isolated vertex.

9) Let G be a graph with 15 vertices and 4 connected components. Prove that G has at least one component with at least 4 vertices.

10) Let G be a graph with 15 vertices and 4 connected components. What is the largest number of vertices that a component of G can have?

11) Find the radius and diameter of the Peterson graph.

12) Let u and v be distinct vertices of a tree T then show that there is precisely one path from u to v in T.

13) Prove that in a group of n people there are two persons having the same number of friends.

14) Let G be a graph without any loops. If for every pair of distinct vertices u and v of G there is precisely one path from u to v in G then show that G is tree.

15) Show that if any five numbers from 1 to 8 are choosen then 2 of them will add up to 9.

16) Let G be a connected graph then show that G is a tree if and only if every edge of G is a bridge.

17) Show that a connected graph G with n vertices has at least $n - 1$ number of edges.

18) Prove that any tree with at least two vertices is a bipartite graph.

19) Let A and B be two finite sets then show that $|A - B| = |A| - |A \cap B|$.

20) Let A and B be two finite sets then show that $|A \cup B| = |A| + |B| - |A \cap B|$.

21) Let A and B be two finite disjoint sets then show that $|A \cup B| = |A| + |B|$.

22) If 101 integers are chosen from the integers 1 to 200 then show that the selection includes two integers such that one of them divides the other.

23) For a bounded distributive lattice L show that complements are unique, if they exist.

24) In a Boolean algebra B prove that, $[a'.(a + b)]' + [b.(b + a')]' + [b.(b' + a)]' = 1$.

25) Show that the dual of a lattice is always a lattice.

26) If p, q, r are the elements of Boolean algebra B then prove that $pqr + pqr' + pq'r + p'qr = pq + qr + rp$.

27) For a bounded distributive lattice L, show that complements are unique, if they exist.

28) Show that $(a+b)' = a' * b'$

29)For every $a, b \in B$ prove that $a + b = 0$ if and only if both $a = 0, b = 0$.

30) For any Boolean algebra B prove that
$(a+b)(b+c)(c+a) = ab + bc + ca$.

31) In a Boolean algebra B prove that if $b + a = c + a$ and
$b + a' = c + a'$ then $b = c$.

32) Show that any linearly ordered set is a lattice.

33) Show that every complete lattice is bounded.

34) Evaluate $1^2 + 2^2 + 3^2 + \cdots + r^2$.

35) Determine the discrete numeric function whose generating function is
$A(z) = \frac{1+z^2}{4-4z+z^2}$.

36) Find the general solution of the difference equation
$a_r + 5a_{r-1} + 6a_{r-2} = 32^r$.

37) Determine the discrete numeric function whose generating function is
$A(z) = \frac{1}{5-6z+z^2}$.

38) Find the general solution of the difference equation
$a_r + 5a_{r-1} + 6a_{r-2} = 32^r - 2r + 1$.

39) Find the generating function of the numeric function $a_r = 2r + 3,\ r \geq 0$.

40) Determine the discrete numeric function whose generating function is
$A(z) = \frac{z^4}{1-2z}$.

41) Find the generating function of the numeric function $a_r = \frac{1}{(r+1)!}\ ; r \geq 0$.

42) A person deposit rupee 100 in a saving account at an interest rate 5% per year compounded annually. Obtain the amount in his account at the end of r years. Sketch the numeric function.

43) Find the generating function of the numeric function $a_r = \frac{r+1}{3^r},\ r \geq 0$.

44) Determine the discrete numeric function whose generating function is
$A(z) = \frac{1}{1-z^3}$.

45) Determine the discrete numeric function whose generating function is
$A(z) = \frac{1+z^2}{4-4z-z^2}$.

4) Long Answer questions

1) a) Let G be a non empty graph with at least two vertices. Show that if G has no odd cycle then G is bipartite.

2) Let G be a simple graph with n vertices and let $\bar{G}$ be its complement.
i) Prove that for each vertex v in G, $d_G(v) + d_{\bar{G}}(v) = n - 1$.
ii) Suppose that G has exactly one even vertex then how many odd vertices does $\bar{G}$ have ?

3) Define self-complementary graph and also prove that if G is the self complementary graph with n vertices then n must be either $4t$ or $4t + 1$, for some integer t.

4) State and prove Handshaking lemma and hence show that in any graph G there is an even number of odd vertices.

5) Given any two vertices of a graph G show that every $u - v$ walk contains $u - v$ path.

6) Let G be a graph with 15 vertices and 4 connected components. Prove that

   a) G has at least one component with at least 4 vertices.

   b) What is the largest number of vertices that a component of G can have?

7) Let T be a tree with n number of vertices then show that T has precisely $n - 1$ edges.

8) State and prove Inclusion-Exclusion principle for three sets.

9) Let T be a tree with n number of vertices then show that T has precisely $n - 1$ edges.

10) Let G be a graph with n number of vertices then show that the following three statements are equivalent.

   $i)$ G is tree.

   $ii)$ G is an acyclic graph with $n - 1$ number of edges.

   $iii)$ G is connected graph with $n - 1$ edges.

11) Let G be a graph with n vertices and q edges and let $w(G)$ denote number of components of G then show that G has at least $n - w(G)$ edges.

12) Among the integers 1 to 1000 how many of them are

   $i)$ not divisible by 3, nor by 5 and nor by 7.

   ii) not divisible by 5 and 7 but divisible by 3.

13) Let G be a connected graph

   i) Let G has 17 edges what is the maximum possible vertices in G?

   ii) If G has 21 vertices, what is the minimum possible number of edges in G?

14) Show that a graph G is connected if and only if it has a spanning tree.

15) Prove that a graph G with n vertices and e edges is unicyclic if and only if G is connected and $n = e$.

16) Prove that for any a, b in a Boolean algebra B following four statements are equivalent

   $i)$ $a * b' = 0.$

   $ii)$ $a + b = b.$

   $iii)$ $a' + b = 1$.

   $iv)$ $a * b = a.$

17) Show that in any finite distributive lattice L, every $a \in L$ can be

written uniquely as the join of irredundant join irreducible elements.

18) Let L be a complemented lattice with unique complements then show that the join irreducible elements of L other than 0 are its atoms.

19) Show that no Boolean algebra can have exactly 3 distinct elements.

20) Prove that for any a, b, c in a Boolean algebra B following four expressions are equal

$i)$ $(a+b)(a'+c)(b+c)$.

$ii)$ $ac + a'b + bc$.

$iii)$ $(a+b)(a'+c)$.

$iv)$ $ac + a'c$

21) Let G be a simple graph. Show that if G is not connected then its complement $\bar{G}$ is connected.

22) Among the integers 1 to 300 how many of them are

$i)$ not divisible by 3, nor by 5.

ii) divisible by 3 but not by 3.

23) Show that there exist a consistent enumeration for any finite poset S.

24) Let L be a lattice. Show that

i) $a \wedge b = a$ if and only if $a \vee b = b$.

ii) The relation $' \leq '$ defined on L as $a \leq b$ if and only if $a \wedge b = a$ or $a \vee b = b$ is a partial order on L.

25) Show that the dual of a lattice is always a lattice.

26) Let a, b, c be any elements in a Boolean algebra B then show that

i) $a + a = a$

ii) $a + 1 = 1$

iii) $a + (a * b) = a$

iv) $a + (b + c) = (a + b) + c$

27) A ball is dropped to the floor from a height of 40 meter suppose that the ball Always rebounds to reach half of the height from which it falls.

i) Let a_r denotes the height it reaches in r^{th} rebound.

ii) Let b_r denote the loss in height during the r^{th} rebound. Sketch the numeric function b also express b_r in terms of a_r.

28) Determine the discrete numeric function whose generating function is

$$A(z) = \frac{7z^2}{(1-2z)(1+3z)} .$$

29) Find the general solution of the difference equation

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r.$$

30) Find the general solution of the difference equation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r.$$

MATHEMATICS
Algebraic Number Theory (MMT-403)

Q.1	**a)**	**Define following terms**
	1)	Irreducible polynomial.
	2)	Symmetric Polynomial
	3)	Trace of α.
	4)	Ring of Integers.
	5)	Quadratic field.
	6)	Algebraic Number.
	7)	Free Abelian Group
	8)	Integral Base.
	9)	Norm of α.
	10)	Unique factorization domain.
	11)	Algebraic integer.
	12)	R-module.
	13)	Conjugates of an algebraic number.
	14)	Discriminant of a basis.
	15)	Cyclotomic Field.
	16)	Principal Ideal Domain
	17)	Unique factorization domain
	18)	Noetherian Domain
	19)	Euclidean Domain
	20)	Maximal ideal
	21)	Prime ideal
	22)	Minkowski's Theorem

	23)	Two Square Theorem
	24)	Four Square Theorem
	25)	The Ramanujan- Nagell Theorem
	26)	Irreducible element
	b)	**Fill in the blanks**
	1)	An algebraic integer is rational number iff it is………………
	2)	Let A and B are ideals of O then g.c.d of ideal A and B is given by G=…………
	3)	The group of units of Z is…………
	4)	Domain D contains x and y such that $x \mid y$ iff ………..
	5)	Let R be a Ring and A is ideal with $\frac{R}{A}$ is field then A is…………. Ideal
	6)	Let A and B be Ideals of O the LCM of A and B is L=…………
	7)	Let R be a Ring and A is ideal with $\frac{R}{A}$ is integral domain then A is …………ideal
	8)	The group of units of $Z[\sqrt{-1}]$ is…………
	9)	Every ideal is …………ideal.
	10)	the ring of integers of $Z[i]$ is Euclidean domain with Euclidean function $\phi(a+ib)=$…….
	11)	The group of units of Z is…………
	12)	In ring R if polynomials $p, q \in R[x]$ then $\partial pq =$……..
	13)	An algebraic integer is rational number iff it is………………
	14)	A non-zero ideal is invertible if PP^{-1}……………
	15)	In domain D, x and y are associates iff…………
	16)	Number of distinct monomorphisms for a field Q(i) is…….
	17)	If K= $Q(\sqrt{5})$ then norm of $2+3\sqrt{5}$ is ………
	18)	If $K = Q(\sqrt{2})$ $then$ $\Delta[1, \sqrt{2}]$ is……
	19)	If K=Q(i) then trace of 2 is………
	20)	$[\mathbb{Q}(\sqrt{2}, \sqrt{3}): \mathbb{Q}]$=………

	21)	Degree of extension [C: R] is ……..
	22)	Minimal polynomial of $\alpha = 2$ is…….
	23)	The G.C.D. of $f(x) = x^3 + x^2 + x + 1$ & $g(x) = x^3 - x^2 + x - 1$
	24)	$[\mathbb{Q}(3^{1/4}) : \mathbb{Q}] =$…….
	25)	Degree of extension [R: R] is ……..
	C)	**Answer the following**
	1)	i) If d $\not\equiv$ 1(mod4) then show that Q ($\sqrt{d}$) has an integral basis of the form $\{1,\sqrt{d}\}$ and discriminant 4d. ii) If d≡1(mod4) then show that Q ($\sqrt{d}$) has an integral basis of the form $\{1, \frac{1}{2} + \frac{\sqrt{d}}{2}\}$ and discriminant d.
	2)	Suppose $\alpha_1,\alpha_2, . . . \alpha_n \in$ O for a Q basis for K if $\Delta[\alpha_1, \alpha_2, \ldots \alpha_n]$is square free then show that $\{\alpha_1,\alpha_2, . . . \alpha_n\}$ is an integral basis
	3)	Prove that every Euclidean Domain is principle ideal domain
	4)	Prove that every Principle ideal domain is Unique factorization domain.
	5)	In ring of integers of K, prove that every prime ideal is a maximal ideal.
	6)	Show that any proper ideal A of a ring of integer O contains a product of prime ideal in O
	7)	Prove that a R-module M is simple if and only if M $\cong \frac{R}{I}$ where I is maximal left ideal of R
	8)	Show that the ring of integer β is a subring of the field of algebraic number $\mathbb{A}$.
	9)	Show that a prime element in D is always irreducible.
	10)	If $\alpha_1,\alpha_2, . . . \alpha_n$ is a basis of K -consisting of integers then prove that the discriminant $\Delta[\alpha_1, \alpha_2, \ldots \alpha_n]$is an rational integer.
	11)	If K = Q($\sqrt{7}$) then the integer K are given by O = Z($\sqrt{7}$) then find norm and trace of Q($\sqrt{7}$)
	12)	Find the number θ such that Q (θ) = Q ($\sqrt{2}, \sqrt{3}$).
	13)	Let R be a ring and A be an ideal of R then show that A is maximal if and

		only if $\frac{R}{A}$ is a field.
	14)	Show that every finite integral domain is a field.
	15)	Find the highest common factor of $p = x^3 + 1 \; and \; q = x^2 + 1 \in \mathbb{Q}[x]$
	16)	Let M_1 and M_2 be sub-modules of R-module M and M = M_1 + M_2 then show that M is direct sum of its sub-modules M_1 and M_2 if and only if $M_1 \cap M_2 = \{0\}$
	17)	Let M = $M_1 \oplus M_2$ then prove that $\frac{M}{M_1} \cong M_2 \; and \; \frac{M}{M_2} \cong M_1$
	18)	In a domain D, in which factorization into irreducible is possible, prove that the factorization is unique iff every irreducible is prime.
	19)	Prove that every non-zero ideal is invertible.
	20)	Show that a R- module M is irreducible if and only if M $\neq \{0\}$ and M is generated by any non zero element in M.
	21)	Let K = Q(θ) be a number field with θ as algebraic number then show that K = Q(α) for some algebraic integer α.
	22)	Prove that the ring of integers O in number field K is Noetherian.
	23)	Find the number θ such that Q (θ) = Q ($\sqrt{2}, \sqrt[3]{3}$).
	24)	Show that the set $\mathbb{A}$ of algebraic number is subfield of the complex field $\mathbb{C}$
	25)	Prove that an algebraic integer is rational number if and only if it is rational integer equivalently $\beta \cap \mathbb{Q} = \mathbb{Z}$.
	26)	Prove that if R be ring and A be an ideal of R then A is prime if and only if $\frac{R}{A}$ is domain.
	27)	Show that an irreducible polynomial over a subfield K of $\mathbb{C}$ has no repeated zeros in $\mathbb{C}$.
	28)	Show that an algebraic number α is an algebraic integer if and only if its minimal polynomial over $\mathbb{Q}$ has coefficients in $\mathbb{Z}$.
	29)	Show that a R-module M is cyclic if and only if M $\cong \frac{R}{I}$ for some ideal I of R.
	30)	Prove that let M be R-module and if $\alpha_1, \alpha_2, \ldots \alpha_n$ be basis for M then M $\cong R^n$ such that $\alpha_i \mapsto (0,0,0,. \; . \; . \; .0,1,0,\ldots 0)$.
	31)	Show that the following conditions are equivalent for an integral domain D i) D is Noetherian ii) D satisfies ascending chain condition

		iii) D satisfies maximal condition.
	32)	If a domain D is Noetherian then prove that factorization into irreducible is possible.
	33)	If K = Q($\sqrt{7}$) then the integer K are given by O = Z($\sqrt{7}$) then find norm and trace of Q($\sqrt{7}$)
	34)	If K is a number field then show that $K = Q(\theta)$ for some algebraic number θ.
	35)	Show that a subgroup of finitely generated abelian group is finitely generated.
	36)	Let Z be a Z submodule with the obvious action. Find all the submodules.
	37)	Show that the coefficient of the field polynomial are rational numbers so that $f_\alpha(t) \in Q(t)$
	38)	Show that a complex number θ is an algebraic integer if and only if the additive group generated by all powers $1, \theta, \theta^2, \ldots$ is finitely generated.
	39)	Find the ring of integers of $Q(\sqrt[3]{5})$
	40)	Show that the ring of integers O of $Q(\zeta)$ is $Z(\zeta)$
	41)	Show that a prime in a domain D is always is irreducible.
	42)	Show that for squarefree d<-11 the ring of integers of $Q(\sqrt{d})$ is not Euclidean.
	43)	State and Prove Ramanujan- Nagell Theorem
	44)	Show that every non-zero ideal of ring of integers O can be written as a product of prime ideals uniquely up to the order of factors.
	45)	If a and b are non-zero ideals of O then show that N(ab)=N(a)N(b)
	46)	Show that factorization of elements of O into irreducibles is unique if and only if every ideal of O is principal
	47)	State and prove two squares theorem.
	48)	State and prove four squares theorem.
	49)	Find the ring of integers of $Q(\sqrt{2}, i)$
	50)	Find the number θ such that Q (θ) = Q ($\sqrt{2}, \sqrt{5}$).

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Paper Name: Fractional Differential Equation(MMT-404)

Question Bank

Q. Define the following terms.

1. Gamma function
2. Beta function
3. Mittag-Leffler function in one parameter
4. GL −derivative of order p
5. RL −integral of order p
6. Gauss product formula for gamma.
7. Legendre formula
8. Mittag-Leffler function in two parameters
9. GL − integral of order p
10. RL −derivative of order p
11. Laplace transformation
12. Capito derivative
13. Caputo integral
14. Relation between gamma and beta function.

Q. Fill in the blanks.

1. $\Gamma(z+4) = \cdots, z \in \mathbb{C}\ and\ Re(z) > 0$
2. $\Gamma\left(\frac{1}{2}\right) = \cdots$
3. For $0 < x < 1, \Gamma(x)\Gamma(1-x) = \cdots$
4. For $p, n \in \mathbb{N}$, function ${f_h}^{(p)}(t)$ is defined as ….
5. For $z \in \mathbb{C}$ and $Re(z) > 0$ and $r \in \mathbb{N}$, $\Gamma(z) = \lim_{r\to\infty}$ ...
6. Let $f \in C[a,b]$ & $p > 0 (p \in \mathbb{R})$. Then for any $t \in [a,b]$, ${}_aD_t^p f(t) = \cdots$.
7. $E_{2,1}(z^2) = \cdots$
8. $\Gamma\left(\frac{3}{2}\right) = \cdots$
9. $\binom{p}{k} = \cdots$
10. $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \cdots$
11. A gamma function has a simple pole at $z = \cdots, (n = 0,1,2,\ldots)$
12. ${}_0D_t^{-\frac{1}{2}} e^{at} = \cdots$
13. ${}_0D_t^{-p} t^{\beta} = \cdots;\ \beta > -1, p > 0, (p \in \mathbb{R})$.
14. For $z, w \in \mathbb{C}$ with $Re(z) > 0, Re(w) > 0$, $\beta(z,w) =$ ____

15. Mittag-Leffler function in one parameter is defined as ____

16. Let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence and $\lim_{k\to\infty}\beta_k = 1$, $\lim_{n\to\infty}\alpha_{n,k} = 0$, $\forall k$ $\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k} = A$, $\forall k$ and $\sum_{k=1}^{n}|\alpha_{n,k}| < K$, $\forall n$ then $\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k =$ ____.

17. GL-fractional integral of order $p > 0$ *and* $f \in C^{m+1}[a,b]$ is expressed as ____

18. GL-derivative of $f(t) = t^{1/2}$ of order $1/2$ is ____.

19. RL-fractional integral of order $p > 0$ is given by ___

20. Let $f(t)$ is integrable function and $p > 0$, $k - 1 \leq p < k$ then ${}_a\mathfrak{D}_t^{-p}({}_a\mathfrak{D}_t^{p}f(t)) =$ ____.

21. Let $f \in C^{m+1}[a,b]$ and $m - 1 < p < m$ $(m \in \mathbb{N})$, p $\in \mathbb{R}$ then $\lim_{p\to m}{}_a\mathfrak{D}_t^{p}f(t) =$ ____.

22. Let $p > 0$(p $\in \mathbb{R}$) then for $p = n(n \in \mathbb{N})$, ${}_a^C D_t^p f(t) =$ ____.

23. The Caputo's fractional derivative of order $1/2$ for $f(t) = t^2$ is ____.

24. The Leibnitz rule for evaluating n^{th} order conventional derivative for the product $\emptyset(t).f(t)$ is ____.

25. The behaviour of fractional derivative far from lower terminal is described by $\lim_{t\to\infty}{}_aD_t^p f(t) =$ ____.

Q. Long Answer Questions.

1. Prove that, $\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\ldots(z+n)}$; $z \in \mathbb{C}, Re(z) > 0$.
2. Show that, $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,2^{-2z+1}\Gamma(2z); 2z \neq 0,-1,-2,\ldots.$
3. Let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence and $\lim_{k\to\infty}\beta_k = 1$, $\lim_{n\to\infty}\alpha_{n,k} = 0$, $\forall k$ $\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k} = A$, $\forall k$ and $\sum_{k=1}^{n}|\alpha_{n,k}| < K$, $\forall n$. Prove that, $\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{n,k}\beta_k = A$.
4. Let $f \in C[a,b]$ & $p > 0(p \in \mathbb{R})$. Then for any $t \in (a,b)$, prove that
$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}f(\tau)d\tau. $$
5. Let $f \in C[a,b]$ & $p > 0(p \in \mathbb{R})$. Then for any $t \in (a,b)$, prove that
$$ {}_aD_t^{p}f(t) = \frac{1}{\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}f(\tau)d\tau. $$
6. Let $f \in C^{m+1}[a,b]$, for some $m \in \mathbb{N}$ and $p > 0(p \in \mathbb{R})$. Then for $t \in (a,b)$, prove that,
$$ {}_aD_t^{-p}f(t) = \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+m+1)}\int_a^t (t-\tau)^{p+m}f^{(m+1)}(\tau)d\tau. $$
7. Let $f \in C^{m+1}[a,b]$, for some $m \in \mathbb{N}$ and $p > 0(p \in \mathbb{R})$. Then for $t \in (a,b)$, prove that,
$$ {}_aD_t^{p}f(t) = \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{-p+m}f^{(m+1)}(\tau)d\tau. $$
8. Evaluate the GL $-$fractional derivative of the power function $(t-a)^{\beta}$ where, p, a, β are real numbers.
9. Let $n \in \mathbb{N}$ and $p > 0(p \in \mathbb{R})$. Prove that,

$${}_aD_t^{p+n}f(t) = \frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) \neq {}_aD_t^p\left(\frac{d^n}{dt^n}f(t)\right).$$

10. Let $m < p < m+1$ $(m \in \mathbb{N}_0)$, let $f(t)$ has $(m+1)$ continuous derivative for $t > 0$. Prove that, ${}_a\mathfrak{D}_t^p f(t) = {}_aD_t^p f(t)$.

11. Let $f(\tau)$ is continuous and integrable on every finite interval (a, t) and may have integrable singularity of order $r < 1$. Then obtain the Cauchy formula for $n-$fold integral of f. Further unify the notation of integer order derivatives and integrals.

12. Let $f \in C[a, b]$ then for any $t \in (c, b)$ prove that, $\lim\limits_{p\to 0} {}_a\mathfrak{D}_t^{-p} f(t) = f(t)$.

13. Let $f(t)$ is continuous for $t \geq a$ then prove that, $RL-$integration of arbitrary real order has the following property,
$${}_a\mathfrak{D}_t^{-q}({}_a\mathfrak{D}_t^{-p} f(t)) = {}_a\mathfrak{D}_t^{-p}\left({}_a\mathfrak{D}_t^{-q} f(t)\right) = {}_a\mathfrak{D}_t^{-p-q}, \text{ for any real } p, q \geq 0.$$

14. Let $k - 1 \leq p < k, k \in \mathbb{N}$. Prove that,
$${}_a\mathfrak{D}_t^{-p}\left({}_a\mathfrak{D}_t^{p} f(t)\right) = f(t) - \sum_{j=1}^{k}\left[{}_a\mathfrak{D}_t^{p-j}\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)}.$$

15. Show that, $RL-$fractional derivative ${}_a\mathfrak{D}_t^p$ and $\frac{d^n}{dt^n}$ do not commute.

16. Show that $RL-$operators do not commute.

17. Let the function $f(t)$ is $(m-1)-$times continuously differentiable in $[a, b]$ and that $f^{(m)}(t)$is integrable in $[a, b]$ and $m - 1 \leq p < m$ $(m \in \mathbb{N})$. Prove that, the condition $[{}_a\mathfrak{D}_t^p f(t)]_{t=a} = 0$ is equivalent to condition
$$f^{(j)}(a) = 0; j = 0,1,2, \dots, m-1.$$

18. Let, $f(t)$ is analytic function. Prove that, for large values of a $i.e.$ $a \to -\infty$,
$${}_aD_t^p f(t) \approx {}_{t-a}D_t^p f(t).$$

19. Let $p > 0, (p \in \mathbb{R})$ and $L\{f(t); s\} = F(s)$. Prove that,
$$L\{ {}_0\mathfrak{D}_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k\left[{}_0\mathfrak{D}_t^{p-k-1} f(t)\right]_{t=0}; n - 1 \leq p < n.$$

20. Let $p > 0, (p \in \mathbb{R})$ and $L\{f(t); s\} = F(s)$. Prove that,
$$L\{ {}_0^CD_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0); n - 1 \leq p < n.$$

21. Using Laplace transforms prove that,
$${}_0D_t^{-\frac{1}{2}} e^{at} = \frac{e^{at}}{\sqrt{a}}\, erf\sqrt{at}.$$

22. Solve Abel's integral of first kind, $\int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau = h(t); 0 < \alpha < 1$ using Laplace transform.

23. If $F_e\{h(t); w\} = H_e(w)$ then find $F_e\{h^{(n)}(t); w\}$.

24. Let $\alpha, \beta, a \in \mathbb{R}$ with $\alpha, \beta > 0$ and $n \in \mathbb{N}_0$ then show that,
$$L\left\{t^{n\alpha+\beta-1} E_{\alpha,\beta}^{(n)}(\pm a t^\alpha); s\right\} = \frac{n!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{n+1}}.\ Re(a) > |a|^{1/\alpha}$$

25. If $L\{f(t); s\} = F(s)$ then prove that, $L\{ {}_0^cD_t^p f(t); s\} =$
$$s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0); (p > 0, p \in \mathbb{R} \text{ } where \text{ } n - 1 \leq p < n\ (n \in \mathbb{N}))$$

26. If $L\{f(t);s\} = F(s)$ then show that, $L\{{}_0\mathfrak{D}_t^p f(t);s\} =$ ____; $(p > 0, p \in \mathbb{R})$

27. Evaluate, the Caputo's fractional derivative of order $3/4$ for $f(t) = t^3$.

28. Evaluate, the Caputo's fractional derivative of order $1/2$ for $f(t) = t^2$.

29. For $n-1 < p < n, \beta \le n-1 (n \in \mathbb{N})$, prove that ${}_a^cD_t^p (t-a)^\beta = 0$.

30. For $n-1 < p < n, \beta > n-1 (n \in \mathbb{N})$, prove that

$${}_a^cD_t^p (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)}(t-a)^{\beta-p}.$$

Q. Short Answer Questions.

1. Show that, $\Gamma(z+1) = z\Gamma(z); z \in \mathbb{C}, Re(z) > 0$.
2. Show that, a gamma function has a simple pole at $z = -n, (n = 0,1,2,\ldots)$.
3. Show that, $\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, z, w \in \mathbb{C}, Re(z), Re(w) > 0$.
4. Let, $p \in \mathbb{N}$ & $f \in C[a,b]$ & $t \in (a,b)$. Prove that ${}_aD_t^p f(t) = \frac{d^p}{dt^p} f(t)$.
5. Let, $p \in \mathbb{N}$ & $f \in C[a,b]$ & $t \in (a,b)$. Prove that,
$${}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh\to t-a}} h^p \sum_{r=0}^{n}(-1)^r \binom{p}{r} f(t-rh) = \frac{1}{(p-1)!}\int_a^t (t-\tau)^{p-1} f(\tau)d\tau.$$
6. Let, $p \in \mathbb{N}$ & $f \in C[a,b]$ & $t \in (a,b)$. Prove that,
$${}_aD_t^{-p} f(t) = \int_a^t d\tau \int_a^t d\tau \ldots \int_a^t f(\tau)d\tau_{(p-times\ integration)}.$$
7. Let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence and $\lim_{k\to\infty} \beta_k = B$, $\lim_{n\to\infty} \alpha_{n,k} = 0$, $\forall k$ $\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k} = A$, $\forall k$ and $\sum_{k=1}^{n} |\alpha_{n,k}| < K$, $\forall n$. Prove that, $\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k = AB$.
8. Let $p < 0 (p \in \mathbb{R}), f \in C[a,b]$ *and* $t \in (a,b)$. Prove that,
$${}_aD_t^q \left({}_aD_t^p f(t) \right) = {}_aD_t^{p+q} f(t) \text{ for any } q \in \mathbb{R}.$$
9. Let $p > 0$ *and* $m \in \mathbb{N}_0$ such that $0 \le m < p < m+1$ and $f \in C^{m+1}[a,b]$ satisfies $f^{(k)} = 0, (k = 0,1,\ldots,m-1)$. Then for any $t \in (a,b)$ prove that, ${}_aD_t^q \left({}_aD_t^p f(t) \right) = {}_aD_t^{p+q} f(t)$ for any $q \in \mathbb{R}$.
10. Let $p, q > 0$, and $m, n \in \mathbb{N}_0$ such that $0 \le m < p < m+1$ and $f \in C^{m+1}[a,b]$ satisfies $f^{(k)} = 0, (k = 0,1,\ldots,r-1)$, where $r = \max\{m,n\}$. Then for any $t \in (a,b)$ prove that,
$${}_aD_t^q \left({}_aD_t^p f(t) \right) = {}_aD_t^p \left({}_aD_t^q f(t) \right) = {}_aD_t^{p+q} f(t) \text{ for any } q \in \mathbb{R}.$$
11. Let $m-1 < p < m$ $(m \in c)$ and $f \in C^{m+1}[a,b]$. For any $t \in (a,b)$ prove that, $\lim_{p\to m-1} {}_a\mathfrak{D}_t^p f(t) = f^{(m-1)}(t)$ and $\lim_{p\to m} {}_a\mathfrak{D}_t^p f(t) = f^{(m)}(t)$.
12. Let $p > 0$ be any real number and $f(t)$ is continuous for $t \ge a$. Prove that, ${}_a\mathfrak{D}_t^p \left({}_a\mathfrak{D}_t^{-p} f(t) \right) = f(t)$.
13. Let $f(t)$ is continuous function, $p, q \ge 0$ be any real numbers and the derivative ${}_a\mathfrak{D}_t^{p-q} f(t)$ exist. Prove that, ${}_a\mathfrak{D}_t^p ({}_a\mathfrak{D}_t^{-q} f(t)) = {}_a\mathfrak{D}_t^{p-q} f(t)$.

14. Let $f(t)$ is continuous function, $p, q \geq 0$ be any real numbers, then prove that ${}_a\mathfrak{D}_t^{-p}\left({}_a\mathfrak{D}_t^{q} f(t)\right) = {}_a\mathfrak{D}_t^{q-p} f(t) - \sum_{j=1}^{k}\left[{}_a\mathfrak{D}_t^{q-j}\right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)}$.

15. Evaluate the RL −fractional derivative of the power function $(t-a)^{\beta}$ where, $p, a, \beta > -1$ are real numbers.

16. Evaluate, ${}_0\mathfrak{D}_t^{\frac{1}{2}}\, {}_0\mathfrak{D}_t^{1} C$, where C is constant.

17. Evaluate, ${}_0\mathfrak{D}_t^{1}\, {}_0\mathfrak{D}_t^{\frac{1}{2}} C$, where C is constant.

18. Evaluate, ${}_0\mathfrak{D}_t^{\frac{3}{2}} C$, where C is constant.

19. Let the function $f(t)$ is $(m-1)$ −times continuously differentiable in $[a, b]$ and that $f^{(m)}(t)$is integrable in $[a, b]$. For every $p(0 < p < m)$, the RL −derivative ${}_a\mathfrak{D}_t^{p} f(t)$ exists and coincide with GL −derivative ${}_aD_t^{p} f(t)$ and if $a < t < b$, then show that following is hold, ${}_a\mathfrak{D}_t^{p} f(t) = {}_aD_t^{p} f(t) =$
$\sum_{j=0}^{m} \frac{f^{(j)}(a)(t-a)^{-p+j}}{\Gamma(-p+j+1)} + \frac{1}{\Gamma(-p+m)} \int_a^t (t-\tau)^{-p+m-1} f^{(m)}(\tau) d\tau.$

20. If for a given continuous function $f(t)$ having integrable derivative, the RL −derivative ${}_a\mathfrak{D}_t^{p} f(t), (0 < p < 1)$ exist and is integrable then for any q such that $0 < q < p$, prove that ${}_a\mathfrak{D}_t^{p} f(t)$ also exist and is integrable.

21. Let $p > 0$ and $n \in \mathbb{N}$ such that $n - 1 < p < n$ and $f(t)$ is a function having $(n + 1)$ −times continuous bounded derivative on $[a, b]$. For any $t \in (a, b)$, prove that $\lim\limits_{p \to n^-} {}_a^CD_t^{p} f(t) = f^{(n)}(t)$ and $\lim\limits_{p \to (n-1)^+} {}_a^CD_t^{p} f(t) = f^{(n-1)}(t) - f^{(n-1)}(a)$

22. Let, $n - 1 < p < n, (n \in \mathbb{N})$ and $m \in \mathbb{N}$. Let $f \in C^{m+n}[a, b]$. Prove that, ${}_a^CD_t^{p}\left({}_a^CD_t^{m} f(t)\right) = {}_a^CD_t^{p+m} f(t) \neq {}_a^CD_t^{m}\left({}_a^CD_t^{p} f(t)\right)$. Further show that the inequality becomes equality only if $f^{(j)} = 0; (j = n, n+1, \dots, m+n-1)$.

23. Let, $n - 1 < p < n, (n \in \mathbb{N})$ and $f \in C^{n}[a, b]$. Then for any $t \in (a, b)$, prove that ${}_a^CD_t^{p} f(a) = 0$.

24. Let λ, μ are constants. Let $p > 0$ be any real numbers and $f(t)$ & $g(t)$ are appropriate functions. Prove that,

$${}_aD_t^{p}\big(\lambda f(t) + \mu g(t)\big) = \lambda\, {}_aD_t^{p} f(t) + \mu\, {}_aD_t^{p} g(t).$$

25. Let λ, μ are constants. Let $p > 0$ be any real numbers and $f(t)$ & $g(t)$ are appropriate functions. Prove that,

$${}_a\mathfrak{D}_t^{p}\big(\lambda f(t) + \mu g(t)\big) = \lambda\, {}_a\mathfrak{D}_t^{p} f(t) + \mu\, {}_a\mathfrak{D}_t^{p} g(t).$$

26. Let λ, μ are constants. Let $p > 0$ be any real numbers and $f(t)$ & $g(t)$ are appropriate functions. Prove that,

$${}_a^CD_t^{p}\big(\lambda f(t) + \mu g(t)\big) = \lambda\, {}_a^CD_t^{p} f(t) + \mu\, {}_a^CD_t^{p} g(t).$$

27. Let $p > 0$ be a real number then evaluate ${}_a^CD_t^{p} K$; where K is constant.

28. Prove that,

$$ {}_a^cD_t^p(t-a)^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-p)}(t-a)^{\beta-p}; n-1<p<n, (n\in\mathbb{N})\ and \\ \qquad\qquad \beta>n-1(\beta\in\mathbb{R}) \\ 0 \qquad ; n-1<p<n, (n\in\mathbb{N})\ and \\ \qquad\qquad \beta\le n(\beta\in\mathbb{N}) \end{cases} $$

29. Evaluate, ${}_0^cD_t^{\frac{1}{2}}(t), {}_0^cD_t^{\frac{1}{3}}(t^2), {}_0^cD_t^{1/2}(t^2)$.

30. Let $p>0, (p\in\mathbb{R})$. Evaluate, ${}_0\mathfrak{D}_t^{-p}e^{at}, a\in\mathbb{C}$

31. Let $p>0, (p\in\mathbb{R})$. Evaluate, ${}_0\mathfrak{D}_t^{p}e^{at}, a\in\mathbb{C}$.

32. Let $p>0, (p\in\mathbb{R})$. Evaluate, ${}_0\mathfrak{D}_t^{-p}sinat$.

33. Let $p>0, (p\in\mathbb{R})$. Evaluate, ${}_0\mathfrak{D}_t^{-p}cosat$.

34. Using Laplace transform prove that for any reals $p,q>0$, ${}_0D_t^{-p}\left({}_0D_t^{-q}f(t)\right) = {}_0D_t^{-q}\left({}_0D_t^{-p}f(t)\right) = {}_0D_t^{-p-q}f(t)$.

35. Using Laplace transforms prove that,

$${}_0D_t^{-p}t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(p+\beta+1)}t^{p+\beta};\ \beta>-1, p>0, (p\in\mathbb{R}).$$

36. Prove that, at $\alpha=1, \beta=2$; $E_{1,2}(z)=\frac{e^z-1}{z}$

37. Prove that, at $\alpha=1, \beta=3$; $E_{1,3}(z)=\frac{1}{z^2}(e^z-z-1)$.

38. Evaluate GL-derivative of $f(t)=(t-3)^6$ of order $7/2$.

39. Evaluate, ${}_aD_t^{5/2}(t+2)^7$.

40. Evaluate, ${}_aD_t^{3/2}(t-9)^6$.

41. Let $p>0(p\in\mathbb{R})$ then evaluate, ${}_a^cD_t^p\ K$ where K is constant.

42. Evaluate, ${}_a^cD_t^{1/2}\ t$.

43. If $L\{f(t);s\}=F(s)$ then find $L\{{}_0\mathfrak{D}_t^pf(t);s\}$.

44. If $M\{f(t);s\}=F(s)$ then find $M\left\{t^\lambda\int_0^\infty \tau^\mu f(t\tau)g(\tau)d\tau;s\right\}$.

45. If $M\{f(t);s\}=F(s)$ then find $M\{t^\alpha f(t);s\}$.